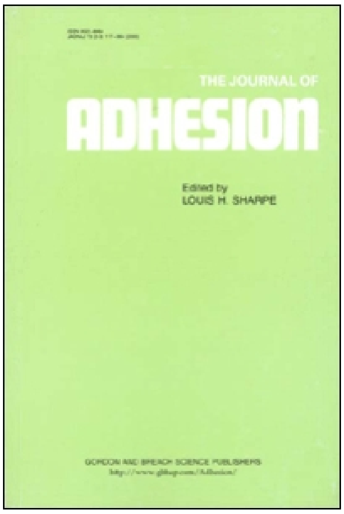

The Journal of Adhesion
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713453635

Equilibrium of Liquid Drops on Thin Plates; Plate Rigidity and Stability Considerations

Martin E. R. Shanahan[a,b]
[a] Laboratoire de Recherches sur la Physico-Chimie des Interfaces, Ecole Nationale Supérieure de Chimie de Mulhouse, MULHOUSE, Cedex, France [b] Centre de Recherches sur la Physico-Chimie des Surfaces Solides, MULHOUSE, France

To cite this Article Shanahan, Martin E. R.(1987) 'Equilibrium of Liquid Drops on Thin Plates; Plate Rigidity and Stability Considerations', The Journal of Adhesion, 20: 4, 261 — 274
To link to this Article: DOI: 10.1080/00218468708074946
URL: http://dx.doi.org/10.1080/00218468708074946

J. Adhesion, 1987, Vol. 20, pp. 261–274
Photocopying permitted by license only

Printed in the United Kingdom

Equilibrium of Liquid Drops on Thin Plates; Plate Rigidity and Stability Considerations

MARTIN E. R. SHANAHAN

Laboratoire de Recherches sur la Physico-Chimie des Interfaces, Ecole Nationale Supérieure de Chimie de Mulhouse, 3, rue Alfred Werner, 68093 MULHOUSE CEDEX, France

and

Centre de Recherches sur la Physico-Chimie des Surfaces Solides, 24, avenue du Président Kennedy, 68200 MULHOUSE, France

(Received September 26, 1985; in final form May 19, 1986)

Following an earlier paper, the equilibrium of an axisymmetric sessile drop resting on a thin, elastic plate has been considered using the minimum free energy hypothesis. A consequence is that although Young's equation is still obeyed at the triple line when considering the true contact angle, the apparent contact angle (*i.e.* with respect to the horizontal) invokes parameters other than simply the interfacial free energies, such as the drop volume and elastic properties of the solid. A quantitative estimate of apparent contact angle dependence on plate thickness has been made in the case of small drops. Finally, consideration of stability, again using the minimum free energy hypothesis, suggests that the axisymmetric configuration may not always be the most likely. This, it is conjectured, may have consequences in cell biology.

KEY WORDS Contact angle; equilibrium; sessile drop; stability; thin elastic plates; Young's equation.

INTRODUCTION

In a recent paper,[1] the present author considered the contact angle equilibrium of axisymmetric, sessile, liquid drops on thin elastic solids. The general differential equations were obtained by a variational treatment, using the hypothesis that the free energy of the system solid/liquid/surrounding fluid will be a minimum at equilibrium, and these were applied to two model solids; a thin plate and a membrane. Whilst the latter was dealt with correctly, the treatment of the thin plate was a little cursory and one of the purposes of this contribution is to correct a misinterpretation of the previous work. It is found that the true contact angle obeys Young's equation.[2] Nevertheless, the apparent contact angle, *i.e.* that measured with respect to the horizontal (or undeformed solid) depends on plate thickness. We shall consider quantitatively modifications to this apparent contact angle in the simplified case in which gravity can be neglected (sufficiently small drops or similar fluid densities). A final section will be devoted to the stability of axisymmetric sessile drops on thin discs. A semi-quantitative analysis shows that, under certain conditions, it may be that the axisymmetric conformation does not represent the minimum in free energy for the system. This fact may facilitate changes in conformation which, it is tentatively suggested, may be of importance in nature, particularly in the context of cell biology.

THEORY

Figure 1 represents an axisymmetric drop of liquid 1, of radius r_0, resting in the centre of a thin, circular, elastic plate, S, of radius a, in the presence of a less dense fiuid 2. For simplicity, the apparent

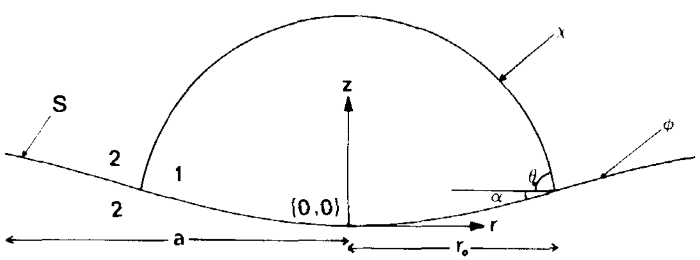

FIGURE 1 Axisymmetric sessile drop of liquid (1) on solid surface (S) in presence of fluid (2) and coordinate system (r, z). χ and ϕ are respectively drop upper surface and solid profiles.

contact angle, θ_0, is considered to be <90° and the usual implicit assumptions of meniscus calculations, such as homogeneity and immiscibility of the phases, are adopted. The free energy of the system (F.E.) is considered to be due to the interfacial contributions, γ_{S1}, γ_{S2} and γ_{12}, the gravitational F.E. associated with the centre of gravity of the drop (that of the thin solid is neglected) and the elastic stored energy:

$$E_T = 2\pi \int_0^{r_0} [(\gamma_{S1} + \gamma_{S2}) r(1 + \phi_r^2)^{1/2} + \gamma_{12} r(1 + \chi_r^2)^{1/2} + \frac{\rho g}{2} r(\chi^2 - \phi^2) + rE_e]\, dr + 2\pi \int_{r_0}^{a} [2\gamma_{S2} r(1 + \phi_r^2)^{1/2} + rE_e]\, dr \quad (1)$$

For a thin, elastic plate, the stored, elastic energy density, E_e, is given by:[3]

$$E_e = \frac{D}{2}\left[\frac{\phi_{rr}^2}{(1 + \phi_r^2)^{5/2}} + \frac{\phi_r^2}{r^2(1 + \phi_r^2)^{1/2}} + \frac{2\nu\phi_r\phi_{rr}}{r(1 + \phi_r^2)^{3/2}}\right] \quad (2)$$

In Eqs. (1) and (2), $\phi(r)$ and $\chi(r)$ represent respectively the solid and liquid upper surface profiles in cylindrical coordinates (r, z), the suffix r has its usual meaning of partial differentiation with respect to r, ρ is the (positive) density difference and g is gravitational acceleration. D and ν are the flexural rigidity and Poisson's ratio of the solid.

At equilibrium, E_T, will be a minimum subject to the constraint of constant drop volume V.

$$V = 2\pi \int_0^{r_0} r(\chi - \phi)\, dr \quad (3)$$

Defining $J = E_T + \lambda V$ where λ is a constant (Lagrange multiplier), we may write:

$$J = \int_0^{r_0} F(r, \phi, \chi, \phi_r, \chi_r, \phi_{rr})\, dr + \int_{r_0}^{a} H(r, \phi_r, \phi_{rr})\, dr \quad (4)$$

Using methods of the calculus of variations, as shown in Ref. 1, and after some algebra, we obtain six relationships pertaining to the conformation at equilibrium of the system. One of these is omitted below since it corresponds simply to the standard capillary equation for the profile of an axisymmetric, sessile drop.

$$\frac{\partial F}{\partial \phi} - \frac{d}{dr}\left(\frac{\partial F}{\partial \phi_r}\right) + \frac{d^2}{dr^2}\left(\frac{\partial F}{\partial \phi_{rr}}\right) = 0 \tag{5}$$

$$\frac{d}{dr}\left(\frac{\partial H}{\partial \phi_r}\right) - \frac{d^2}{dr^2}\left(\frac{\partial H}{\partial \phi_{rr}}\right) = 0 \tag{6}$$

$$\left[\frac{\partial F}{\partial \chi_r} + \frac{\partial F}{\partial \phi_r} - \frac{d}{dr}\left(\frac{\partial F}{\partial \phi_{rr}}\right) - \frac{\partial H}{\partial \phi_r} + \frac{d}{dr}\left(\frac{\partial H}{\partial \phi_{rr}}\right)\right]_{r=r_0} = 0 \tag{7}$$

$$\Bigg[F - \frac{\partial F}{\partial \chi_r}\cdot \chi_r - \frac{\partial F}{\partial \phi_r}\cdot \phi_r + \frac{d}{dr}\left(\frac{\partial F}{\partial \phi_{rr}}\right)\cdot \phi_r - \frac{\partial F}{\partial \phi_{rr}}\cdot \phi_{rr}$$
$$- H + \frac{\partial H}{\partial \phi_r}\cdot \phi_r - \frac{d}{dr}\left(\frac{\partial H}{\partial \phi_{rr}}\right)\cdot \phi_r + \frac{\partial H}{\partial \phi_{rr}}\cdot \phi_{rr}\Bigg]_{r=r_0} = 0 \tag{8}$$

$$\left[\frac{\partial F}{\partial \phi_{rr}} - \frac{\partial H}{\partial \phi_{rr}}\right]_{r=r_0} = 0 \tag{9}$$

In Ref. 1, it was incorrectly stated that Eq. (9) was of no use. Clearly for a plate, both ϕ and ϕ_r must be continuous at r_0, otherwise infinitely sharp folding, or breaking of the material is implied. However, relation (9) when applied to the above integral definitions, allows us to infer that ϕ_{rr} is also continuous at r_0.

Both relations (7) and (8) were incompletely interpreted in Ref. 1. A correct evaluation of these using the above expressions (1), (2) and (3) leads respectively to:

$$\frac{(\gamma_{S1} - \gamma_{S2})\cdot \phi_r(r_0)}{[1+\phi_r^2(r_0)]^{1/2}} + \frac{\gamma_{12}\cdot \chi_r(r_0)}{[1+\chi_r^2(r_0)]^{1/2}} - \frac{D\cdot[\phi_{rrr}^{(1)}(r_0) - \phi_{rrr}^{(2)}(r_0)]}{[1+\phi_r^2(r_0)]^{5/2}} = 0 \tag{10}$$

and:

$$\frac{(\gamma_{S1} - \gamma_{S2})}{[1+\phi_r^2(r_0)]^{1/2}} + \frac{\gamma_{12}}{[1+\chi_r^2(r_0)]^{1/2}} + \frac{D\cdot \phi_r(r_0)\cdot[\phi_{rrr}^{(1)}(r_0) - \phi_{rrr}^{(2)}(r_0)]}{[1+\phi_r^2(r_0)]^{5/2}} = 0 \tag{11}$$

where superfixes (1) and (2) represent the values taken by the functions at r_0 when approaching respectively from inside and from outside the drop. Although ϕ_{rr} is continuous at r_0 [by Eq. (9)], ϕ_{rrr} is not necessarily. This is the cause of the error introduced in the earlier paper where the terms in flexural rigidity were wrongly assumed always to be zero (Eqs. (11) and (9) in Ref. 1).

Both Eqs. (10) and (11) can be simplified by noting that $\chi_r(r_0) = -\tan\theta_0$ and $\phi_r(r_0) = \tan\alpha$ where θ_0 and α are the inclinations of the liquid and the solid at the contact line (see Figure 1). In addition it is clear that:

$$\frac{d}{dr}\left(\frac{1}{R}\right) = \frac{d}{dr}\left(\frac{\phi_{rr}}{[1+\phi_r^2]^{3/2}}\right) = \frac{\phi_{rrr}}{(1+\phi_r^2)^{3/2}} - \frac{3\phi_r \cdot \phi_{rr}^2}{(1+\phi_r^2)^{5/2}} \tag{12}$$

where R represents the local radius of curvature of the function $\phi(r)$.

Using these facts, Eqs. (10) and (11) can be reduced to:

$$\gamma_{12} \cdot \sin\theta_0 + (\gamma_{S2} - \gamma_{S1}) \cdot \sin\alpha + D \cdot \cos^2\alpha \left[\frac{d}{dr}\left(\frac{1}{R^{(1)}}\right)\bigg|_{r_0} - \frac{d}{dr}\left(\frac{1}{R^{(2)}}\right)\bigg|_{r_0}\right] = 0 \tag{13}$$

$$\gamma_{12} \cdot \cos\theta_0 - (\gamma_{S2} - \gamma_{S1}) \cdot \cos\alpha + D \cdot \sin\alpha\cos\alpha \left[\frac{d}{dr}\left(\frac{1}{R^{(1)}}\right)\bigg|_{r_0} - \frac{d}{dr}\left(\frac{1}{R^{(2)}}\right)\bigg|_{r_0}\right] = 0 \tag{14}$$

Equation (14) can be taken as the modified Young equation for the apparent contact angle equilibrium on a thin, elastic solid modelled by thin plate theory. Note that, as before, it reduces to Young's equation for an undeformable solid.

Combination of Eqs (13) and (14) leads to:

$$\gamma_{12}\cos(\theta_0 + \alpha) + \gamma_{S1} - \gamma_{S2} = 0 \tag{15}$$

$$\gamma_{12}\sin(\theta_0 + \alpha) + D\cos\alpha \left[\frac{d}{dr}\left(\frac{1}{R^{(1)}}\right)\bigg|_{r_0} - \frac{d}{dr}\left(\frac{1}{R^{(2)}}\right)\bigg|_{r_0}\right] = 0 \tag{16}$$

Clearly $(\theta_0 + \alpha)$ is the true contact angle measured between the solid surface and the tangent to the γ_{12} interface. Young's equation is therefore obeyed [Eq. (15)]. Nevertheless, in practice, measurement of $(\theta_0 + \alpha)$ maybe difficult whereas the apparent contact angle, θ_0, is readily found. Equation (16) perhaps answers Bikerman's query of long standing,[4,5] *i.e.* what balances the liquid surface tension component perpendicular to the solid surface at the contact line? In the present case, this component is $\gamma_{12}\sin(\theta_0 + \alpha)$

and Eq. (16) shows how this is equilibrated by an elastic, flexural term due to the deformation of the thin solid. Nevertheless, as previously stated in Ref. 1, no real material will behave perfectly as a mathematically thin, elastic plate. In all probability, a slight ridge will develop in the solid along the contact line, but consideration of this aspect represents a topic outside the context of the present paper.[6]

The last point to be made about Eqs (13), (14) and (16) is that all three contain a term invoking changes in radii of curvature of the solid at the contact line. Evaluation of this term would require a detailed knowledge of the boundary conditions of the problem in the general case, *i.e.* methods of clamping the solid, drop size, etc. . .

INFLUENCE OF PLATE RIGIDITY

Equations (5) and (6) above may be applied to the relevant expressions for free energy and constant volume and two fourth order differential equations arise describing the profile of the solid surface respectively beneath the axisymmetric sessile drop and outside, *i.e.* for $a \geqslant r \geqslant r_0$. These equations, given in Ref. 1, are rather complicated and very probably insoluble analytically. Nevertheless, both can be solved approximately, the former using perturbation theory and the latter by reduction to a form of Euler's differential equation, when terms of small magnitude can be neglected. Under these circumstances, to a first approximation, the profile under the drop, ϕ, is circular and that outside, now to be referred to as Φ, is logarithmic when the plate is unclamped. Under conditions in which gravity is negligible (small drops or similar fluid densities), the upper drop surface profile is also circular. We may then write the three profiles as:

$$\phi \simeq R - (R^2 - r^2)^{1/2} \tag{17}$$

$$\Phi \simeq \frac{r_0^2}{(R^2 - r_0^2)^{1/2}} \cdot \ln r + \text{constant} \tag{18}$$

$$\chi \simeq (P^2 - r^2)^{1/2} + R - (P^2 - r_0^2)^{1/2} - (R^2 - r_0^2)^{1/2} \tag{19}$$

where R and P represent the radii of curvature of the lower and upper drop surfaces. Since the gravitational F.E. term is neglected, substitution of the above and various derivatives into Eqs (1), (2) and (3) leads to an expression for the function $J = E_T + \lambda V$. Assuming that $a \gg r_0$, this may be integrated to give to a first approximation:

$$\begin{aligned} J/2\pi \simeq & (\gamma_{S1} + \gamma_{S2})[R - (R^2 - r_0^2)^{1/2}] \cdot R \\ & + \gamma_{S2}(a^2 - r_0^2) + \gamma_{12}[P - (P^2 - r_0^2)^{1/2}] \cdot P \\ & + \frac{D(1+\nu)}{R} \cdot [R - (R^2 - r_0^2)^{1/2}] + \frac{D}{2} \cdot \frac{r_0^2}{(R^2 - r_0^2)} \cdot (1 - \nu) \\ & + \lambda \left\{ \frac{1}{3}[P^3 + R^3 - (P^2 - r_0^2)^{3/2} - (R^2 - r_0^2)^{3/2}] \right. \\ & \left. - \frac{r_0^2}{2} \cdot [(P^2 - r_0^2)^{1/2} + (R^2 - r_0^2)^{1/2}] \right\} \end{aligned} \tag{20}$$

The problem is now to determine the four unknowns r_0, R, P and λ. This can be done using the differential calculus and the method of Lagrange multipliers. For J to be a minimum, apart from V being constant, we must have:

$$\left. \begin{aligned} \frac{\partial J}{\partial P} &= 0 \quad \text{(a)} \\ \frac{\partial J}{\partial R} &= 0 \quad \text{(b)} \\ \frac{\partial J}{\partial r_0} &= 0 \quad \text{(c)} \end{aligned} \right\} \tag{21}$$

Evaluation of Eq. (21) (a) leads simply to:

$$\lambda = -2\gamma_{12}/P \tag{22}$$

Equation (21) (b), using Eq. (22), leads to:

$$\gamma_{S1} + \gamma_{S2} - \frac{\sin\theta_0}{\sin\alpha} \cdot \gamma_{12} + \frac{D r_0^2}{R^2[R - (R^2 - r_0^2)^{1/2}]^2} \cdot \left[1 + \nu + \frac{(1-\nu)R^3}{(R^2 - r_0^2)^{3/2}} \right] = 0 \tag{23}$$

and similar treatment of Eq. (20) using Eq. (21) (c) produces:

$$\gamma_{S1} + \gamma_{S2}(1 - 2\cos\alpha) + \gamma_{12}\cos(\theta_0 + \alpha) + D\left[\frac{(1+\nu)}{R^2} + \frac{(1-\nu)R}{(R^2 - r_0^2)^{3/2}}\right] = 0 \quad (24)$$

In both Eqs (23) and (24), some simplification of the γ terms has been made using elementary trigonometry.

In principle, eqs (22), (23) and (24) can be solved simultaneously together with the constant volume condition, but in practice it is simpler to opt for constant r_0 and let the volume vary. The following solution assumes $R \gg r_0$, *i.e.* that the solid is only relatively slightly deformed from its planar aspect. Under these conditions, Eq. (23) simplifies to:

$$\gamma_{S1} + \gamma_{S2} - \frac{\sin\theta_0}{\sin\alpha}\cdot\gamma_{12} + \frac{8D}{r_0^2} \simeq 0 \quad (25)$$

Simplification of eq. (24) and use of Eq. (25) gives:

$$\gamma_{S1} - \gamma_{S2} + \gamma_{12}\cos\theta_0 - \frac{6\,D\sin^2\alpha}{r_0^2} \simeq 0 \quad (26)$$

Elimination of θ_0 leads to a quadratic equation in $\sin^2\alpha$. We can thus examine the variation of angle α as a function of the variables of the system and then return to calculate θ_0, with the proviso that α must be sufficiently small for the above approximations to hold reasonably accurately.

It is interesting to note at this stage that Eq. (25) reduces to Laplace's equation for a liquid lens of "double" lower interface, $(\gamma_{S1} + \gamma_{S2})$, if $D \to 0$. Analogously, Eq. (26) reduces to Young's equation if $\alpha \to 0$.

The above analysis has been applied to the case of a small drop (radius $r_0 = 0.1$ cm) of 1-bromonaphthalene resting on a thin mica sheet. Young's modulus, E, was taken to be 10 GPa and Poisson's ratio, ν, 1/3. The values of γ_{12} and γ_{S2} were taken to be respectively 44.6 and 120 mJ.m^{-2}.[7]

Assuming that 1-bromonaphthalene is an essentially apolar liquid, and using Fowkes' relationship for interfacial tensions employing the geometric mean,[8] we find γ_{S1} to be 91.5 mJ.m^{-2}. Since the largest possible value of α will correspond to zero flexural rigidity of the plate, we may employ Neumann's triangle[9] to establish the upper limit (taking into account that the lower interface is "double" since there are two sides corresponding to γ_{S1} and to γ_{S2}). From this we find $\alpha = 8.7°$ and $\theta_0 = 46.1°$. This assumption of $R \gg r_0$, or relatively little plate deformation, will therefore be reasonably valid for all but the very thinnest of mica sheets. The above analysis was therefore employed to consider the variations of both α and θ_0 as a function of thickness of the mica. The results are shown in Figure 2. The lower abscissa refers to actual plate thickness, t, and the upper to the value of the flexural rigidity, D. It can be seen that the effect of plate bending is practically negligible for thicknesses greater than about 10 μm where the solutions rapidly tend to those of the undeformable solid ($\alpha = 0°$, $\theta_0 = 50.3°$), but that below this value α increases and θ_0 decreases tending towards the limiting values given by Neumann's triangle for the liquid lens corresponding to the plate of $t \rightarrow 0$, but retaining both γ_{S1} and γ_{S2} interfaces. Values of α and

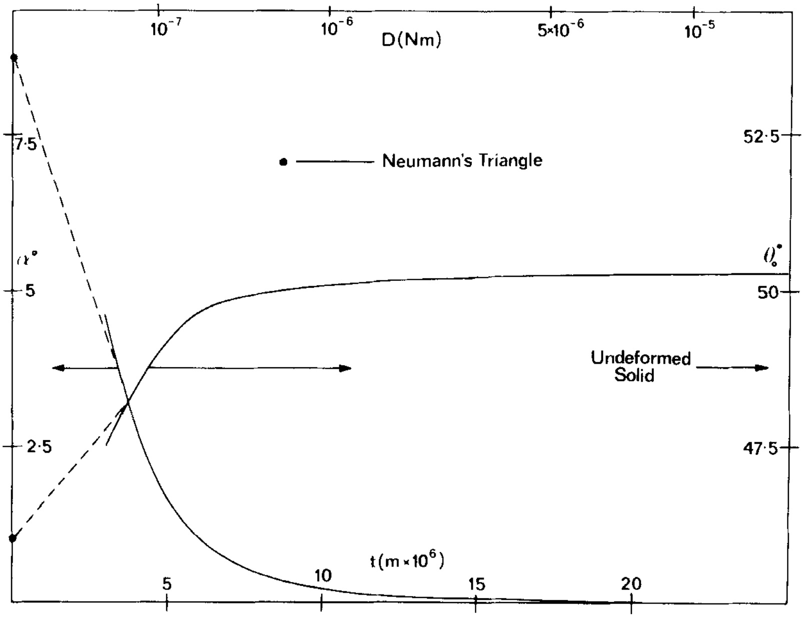

FIGURE 2 Angles α and θ_0 as a function of plate thickness and flexural rigidity, for drop of 1-bromonaphthalene on mica (see text for details).

θ_0 corresponding to $t \leqslant 3\,\mu$m have been extrapolated since the above simplified analysis starts to introduce non-negligible error in this range.

It may be seen in this example that the modification to Young's equation is very small when considering practical, every day values of apparent contact angle, θ_0. The error involved in neglecting α is minute. Nevertheless, the effect of deformation of the solid can be significant on a microscopic scale, and this, it is conjectured, may be of importance in cellular biology. The following section will attempt to elucidate this aspect semi-quantitatively.

AXISYMMETRIC DROP STABILITY

The above, and the previous paper,[1] have analysed the system plate/drop considering that it is axisymmetric. It is assumed that, at equilibrium, the final configuration adopted will be that shown in Figure 3 (a) where the initially flat, thin, plate has a slight axisymmetric depression in its centre accommodating the liquid drop. That axial symmetry be the true state of equilibrium for either a liquid drop on an undeformable solid, or a liquid lens, *i.e.* a drop resting on an immiscible, denser liquid substrate, is clear. However, in the intermediate case of a thin solid, this is not so clear. The

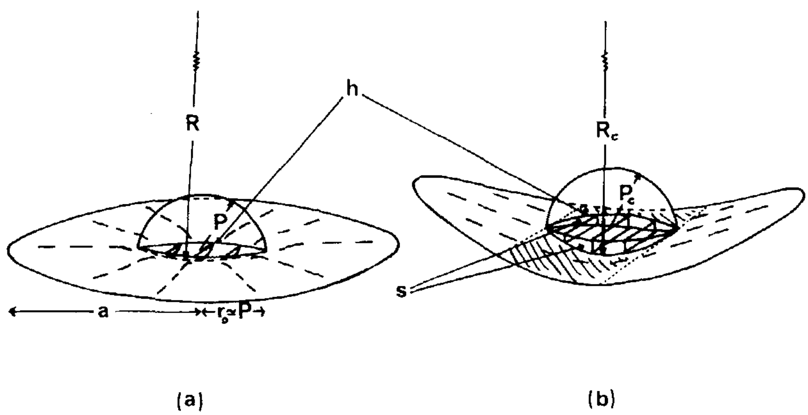

FIGURE 3 Schematical representation of drop on thin disc: (a) axisymmetric configuration, (b) cylindrical configuration (see text for details).

semi-quantitative argument below is intended to illustrate this point.

We shall on one hand consider the total F.E. of a system plate/drop assuming an axisymmetric configuration and, on the other hand, a possible different configuration; that of a similar drop resting in a channel in the (still circular) plate produced by cylindrical bending. This latter is shown schematically in Figure 3(b). The two systems will now be referred to respectively as (a) and (b). Both cases are considered in the absence of a gravitational term. Clearly the configuration adopted by the system at equilibrium will be that which minimises its overall F.E.. There is no guarantee that (b) will represent the conformation of absolute minimum F.E., but if its F.E. is inferior to that of (a), then it will be a more likely, *i.e.* more stable, configuration and, more important, it will show that the axisymmetric form is not the one adopted naturally. The argument hinges on the fact that the logarithmic form adopted in (a) by the plate outside the drop, function Φ, implies that there is strain energy associated with the entire deformed plate, whereas, apart from the cylindrically bent channel in (b), the rest of the plate is unstrained. The strain energy of (b) can then be less than (a). Nevertheless, the hypothetical change of (a) to (b) also introduces potential changes in F.E. due to γ terms. If any such increase is less than the decrease in strain energy, (b) will present a lower total F.E. This resumes the argument qualitatively.

We now consider that γ_{S1} and γ_{S2} are equal. This implies that (i) θ_0 will be close or equal to 90°, (ii) $r_0 \simeq P$ and (iii) that any changes in the surface area of the solid disc in contact with the liquid are not associated with changes in total F.E. We retain the same nomenclature as before when referring to the axisymmetric case and introduce the same symbols but with suffix c for equivalent variables in the cylindrically bent case (see Figure 3).

If the radii of curvature R and R_c are large and the volume of the drop, V, is to stay constant during the hypothetical transformation from (a) to (b), it is readily shown that:

$$V \simeq \pi P^3\left[\frac{2}{3}+\frac{P}{4R}\right] \simeq \pi P_c^3\left[\frac{2}{3}+\frac{3P_c}{8R_c}\right] \tag{27}$$

In each of these expressions, the first term in brackets refers to the volume of the drop above the horizontal marked h in Figure 3, and the second term that underneath. Defining $P_c = P(1-\varepsilon)$ and $R_c = R(1-\delta)$ where ε and δ are increments, Eq. (27) leads to the fact that, to first order:

$$\delta \simeq \left[4 + \frac{16R}{3P}\right]\varepsilon - \frac{1}{3} \tag{28}$$

The F.E. of (a) pertaining *only* to γ_{12} and D (since gravity is neglected and changes in contact area are of no consequence, γ_{S1} equalling γ_{S2}) can be written [*cf.* Eq. (20)]:

$$E' \simeq 2\pi\left\{\gamma_{12}P^2 + D \cdot \left[\frac{(1+\nu)}{R}\left[R - (R^2 - P^2)^{1/2}\right] + \frac{P^2(1-\nu)}{2(R^2 - P^2)}\right]\right\} \tag{29}$$

Consider now the equivalent F.E. for case (b). The extra surface area of the γ_{12} interface in this configuration is that shown by s in Figure 3(b) and resembling two dentures. Its value is to a first approximation $\pi P_c^3/2\,R_c$. The F.E. due to plate bending will involve only the cylindrical channel shown shaded in Figure 3(b), and since only one radius of curvature is invoked (the other being infinite), the expression for E_e in Eq. (2) only contains the first term in brackets. Evaluation of the stored elastic energy along the whole channel length (both beneath and outside the drop) assuming constant radius of curvature, R_c, is then simple. We can thus obtain an expression for E_c', the F.E. involving only γ_{12} and D:

$$E_c' \simeq \pi P_c^2\gamma_{12}\left[2 + \frac{P_c}{2R_c}\right] + \frac{2aD}{R_c} \cdot \sin^{-1}\left(\frac{P_c}{R_c}\right) \tag{30}$$

Clearly if $E' > E_c'$, the cylindrical configuration (b) will be more stable than the axisymmetric case (a). Using relation (28) in Eq. (30) and simplifying (29) and (30) further using the assumption of $R \gg P$ and $R_c \gg P_c$, it can be shown that the condition $E' > E_c'$ is equivalent to:

$$\frac{2D}{R^2}\left\{P + \frac{P^3}{8R^2}(5-3\nu) - \frac{a}{3\pi}\right\} + \left\{\gamma_{12}P\left(\frac{4}{3} - \frac{P}{2R}\right) - \frac{2aD}{\pi R}\left(\frac{7}{R} + \frac{32}{3P}\right)\right\} \cdot \varepsilon \gtrsim \frac{P^2\gamma_{12}}{3R} \tag{31}$$

Under conditions in which ε is very small and the interfacial tension γ_{12} is negligible, relation (31) can be simplified much further still to obtain:

$$P \geqslant \frac{a}{3\pi} \simeq 0.1a \tag{32}$$

Remembering that in the above derivation, it was assumed that $P \simeq r_0$, we can conclude that if the drop diameter is greater than *ca.* 10% of the plate diameter, it may be possible to have the cylindrical conformation representing a lower state of F.E. than the axisymmetric case and therefore a more stable configuration. The above analysis is necessarily a little sketchy since a rigorous, analytical treatment of the non-axisymmetric situation would be very difficult, if not impossible, mathematically. However, the point to be made is that sessile drops posed on thin plates are not inevitably always going to lead to an axisymmetric configuration, although our hypothetical cylindrically bent system does not necessarily represent the most favourable alternative. It is simply a mathematical model. Factors favouring the non-axisymmetric case will be non-axisymmetric clamping, low but non-zero plate ragidity, similarity in values of γ_{S1} and γ_{S2} and a low value of γ_{12}. Many of these factors are likely to be met in biological systems at a cellular level. Since the difference in F.E. of systems in the axisymmetric and non-axisymmetric conformations will generally be small, this should facilitate the transformation between them. It is therefore conjectured that a suitable combination of values of the parameters involved will tend to facilitate mobility at a cellular level. Changes in shape of cell walls will be rendered easier. Fanciful extrapolation leads one to consider that an eventual tool for reducing cell mobility (in the spread of cancer?) could be the successful modification of D or γ terms rendering cell walls less manoeuvrable. This, however, is sheer conjecture and best left to the cell biologist to consider seriously.

CONCLUSION

The system consisting of an axisymmetric sessile drop centred on a circular, thin, elastic plate in the presence of a second, immiscible fluid has been considered from three aspects. In the first, the equations describing Young's relationship for contact angle equilibrium have been rederived correcting an error in interpretation of an earlier paper. Secondly, a quantitative analysis has been effected to assess the variations in apparent contact angle due to deformation of the solid substrate. Finally, a semi-quantitative analysis of the stability of the axisymmetric configuration has been undertaken. It shows that under certain conditions, it may be that axisymmetry does not represent the most stable conformation of the drop/plate system. A conjectured consequence is the rôle this instability may play in cell biology.

REFERENCES

1. M. E. R. Shanahan, *J. Adhesion* **18** (4), 247 (1985).
2. T. Young, *Phil. Trans. R. Soc. London* **95,** 65 (1805).
3. S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity,* 3rd Ed., (Mc Graw-Hill, New York, 1970), pp. 246–7.
4. J. J. Bikerman, *J. Phys. Chem.* **63,** 1658 (1959).
5. J. J. Bikerman, *Physical Surfaces,* (Academic Press, London, 1970), Ch. 6.
6. M. E. R. Shanahan and P. G. de Gennes, *Compt. Rend. Acad. Sci.,* Paris, 302, série II, no. 8, 517 (1986).
7. J. Schultz, K. Tsutsumi and J. B. Donnet, *J. Colloid Interface Sci.* **59,** 272 and 277 (1977).
8. F. M. Fowkes, *Treatise on Adhesion and Adhesives,* Vol. 1, R. L. Patrick, Ed., (Marcel Dekker, New York, 1967), Ch. 9.
9. F. Neumann, *Vorlesungen über die Theorie der Capillarität,* (B. G. Teubner, Leipzig, 1894).